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Black Hole Attractor Varieties and Complex Multiplication[◇]

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Abstract

Black holes in string theory compactified on Calabi-Yau varieties a priori might be expected to have moduli dependent features. For example the entropy of the black hole might be expected to depend on the complex structure of the manifold. This would be inconsistent with known properties of black holes. Supersymmetric black holes appear to evade this inconsistency by having moduli fields that flow to fixed points in the moduli space that depend only on the charges of the black hole. Moore observed in the case of compactifications with elliptic curve factors that these fixed points are arithmetic, corresponding to curves with complex multiplication. The main goal of this talk is to explore the possibility of generalizing such a characterization to Calabi-Yau varieties with finite fundamental groups.

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1 Introduction

Arithmetic considerations have taken center stage in algebraic geometry during the past two or three decades. In particular the search for the still somewhat elusive concept of motives has motivated much of current research. Few of these more modern developments have had any impact on physics, even though it has been clear for almost two decades that algebraic geometry is central to the understanding of string theory. This is perhaps surprising because many of the number theoretic and arithmetic results are closely linked to powerful analytic tools. Recently however, arithmetic structures have been used to address a variety of problems in string theory, such as the problem of understanding aspects of the underlying conformal field theory of Calabi-Yau manifolds [20], the nature of black hole attractor varieties [18], and the behavior of periods under reduction to finite fields [4].

In this paper we describe some further developments and generalizations of some of the observations made by Moore in his analysis of the arithmetic nature of the so-called black hole attractor varieties [17]. The attractor mechanism [10, 23, 11, 12] describes the radial evolution of vector multiplet scalars of spherical dyonic black hole solutions in $N = 2$ supergravity coupled to abelian vector multiplets. Under particular regularity conditions the vector scalars flow to a fixed point in their target space. This fixed point is determined by the charge of a black hole, described by a vector ω in the lattice Λ of electric and magnetic charges of the $N = 2$ abelian gauge theory. If the $N = 2$ supergravity theory is derived from a type IIB string theory compactified on a Calabi-Yau space, the vector multiplet moduli space is described by the moduli space $\mathcal{M}$ of complex structures of X, and the dyonic charge vector takes values in the lattice $\Lambda = \mathrm{H}^3(\mathrm{X}, \mathbb{Z})$.

One of the crucial observations made by Moore is that in the context of simple toroidal product varieties, such as the triple product of elliptic curves E^3, or the product of the K3 surface and an elliptic curve K3$\times E$, the attractor condition determines the complex moduli τ of the tori to be given by algebraic numbers in a quadratic imaginary field $\mathbb{Q}(\sqrt{D})$, $D < 0$. This is of interest because for particular points in the moduli space the elliptic curves exhibit additional symmetries, that is, they admit so-called complex multiplication (CM). For compactifications

with such toroidal factors Moore's analysis then appears to indicate a strong link between the 'attractiveness' of varieties in string theory and their complex multiplication properties. We will briefly review Moore's observations in Section 2.

Calabi-Yau varieties with elliptic factors are very special because they have infinite fundamental group, a property not shared by Calabi-Yau manifolds in general. Other special features of elliptic curves are not present in general either. In particular Calabi-Yau spaces are not abelian varieties and they do not, in any obvious fashion, admit complex multiplication. Hence it is not clear how Moore's observations can be generalized. It is this problem which is addressed in [17]. In order to formulate such a generalization we adopt a cohomological approach and view the modular parameter of the elliptic curve as part of the primitive cohomology. In the case of elliptic curves E this is simply a choice of point of view because there exists an isomorphism between the curve itself and its Jacobian defined by $J(E) = \mathrm{H}^1(E,\mathbb{C})/\mathrm{H}^1(E,\mathbb{Z})$ described by the Abel-Jacobi map $j : E \to J(E)$. These varieties are abelian.

The Jacobian variety of elliptic (and more general) curves has a natural generalization to higher dimensional varieties defined by the intermediate Jacobian of Griffiths. It would be natural to use Griffiths' construction in an attempt to generalize the elliptic results described above. In general, however, the intermediate Jacobian is not an abelian variety and does not admit complex multiplication. For this reason we will proceed differently by constructing a decomposition of the intermediate cohomology of the Calabi-Yau variety. We then use this decomposition to formulate a generalization of the concept of complex multiplication of black hole attractor varieties. To achieve this we formulate complex multiplication in this more general context by analyzing in some detail the cohomology group $\mathrm{H}^3(\mathrm{X})$ of weighted Fermat hypersurfaces X.

The paper is organized as follows. In Section 4 we briefly review the necessary background of abelian varieties. In Section 5 we show how abelian varieties can be derived from Calabi-Yau hypersurfaces by showing that the cohomology of such varieties can be constructed from the cohomology of curves embedded in these higher dimensional varieties. This leads us to abelian varieties defined by the Jacobians of curves. Such abelian varieties do not, in general,

admit complex multiplication. However, it is known that Jacobians of ordinary projective Fermat curves split into abelian factors which do admit complex multiplication. We briefly describe this construction and demonstrate that this property generalizes to Brieskorn-Pham curves. Combining these results shows that we can consider the complex multiplication type of Calabi-Yau varieties as determined by the CM type of their underlying Jacobians.

When these results are applied to describe the complex multiplication type of a particularly simple black hole attractor variety it emerges that its complex multiplication leads precisely to the field determined by its periods. It is in fact not completely unexpected that we might be able to recover the field of periods by considering the complex multiplication type. The reason for this is a conjecture of Deligne [7] which states that the field determined by the periods of a critical motive is determined by its L-function. Because Deligne's conjecture is important for our general view of the issue at hand we briefly describe this conjecture in Section 3 in order to provide a broader perspective. It is important to note that Deligne's conjecture is in fact a theorem in the context of pure projective Fermat hypersurfaces [3], but has not been proven in the context of weighted hypersurfaces. Our results in essence can be viewed as support of this conjecture even in this more general context. In Section 2 we briefly review the physical setting of black hole attractors in type IIB theories, as well as Moore's solution of the K3 $\times$ E solution of the attractor equations. In Section 7 we summarize our results and indicate possible generalizations.

2 Attractor Varieties

2.1 Compactified type IIB string theory

We consider type IIB string theory compactified on Calabi-Yau threefold varieties. The field content of string theory in 10D space X_{10} splits into two sectors according to the boundary conditions on the world sheet. The Neveu-Schwarz fields are given by the metric $g \in \Gamma(X_{10}, \mathrm{T}^*\mathrm{X}_{10} \otimes \mathrm{T}^*\mathrm{X}_{10})$, an antisymmetric tensor field $B \in \Gamma(X_{10}, \Omega^2))$ and the dilaton scalar $\phi \in C^\infty(X_{10}, \mathbb{R})$. The Ramond sector is spanned by even antisymmetric forms

$A^p \in \Gamma(X_{10}, \Omega^p)$ of rank p zero, two, and four. Here $\Omega^p \longrightarrow X$ denotes the bundle of p-forms over the variety X.

In the context of the black hole solutions considered in [10] the pertinent sectors are given by the metric and the five-form field strength $\mathbf{F}$ of the Ramond-Ramond 4-form A^4. The metric is assumed to be static, spherically symmetric, asymptotically Minkowskian, and should describe extremally charged black holes, leading to the ansatz

$$ds^2 = -e^{2U(r)} dt \otimes dt + e^{-2U(r)}(dr \otimes dr + r^2 \sigma_2), \tag{1}$$

where r is the spatial three dimensional radius, σ_2 is the 2D angular element, and the asymptotic behavior is encoded via $e^{-U(r)} \to \infty$ for $r \to \infty$. The ten-dimensional five-form $\mathbf{F}$ leads to a number of different four-dimensional fields, the most important in the present context being the field strengths F^L of the four dimensional abelian fields, the number of which depends on the dimension of the cohomology group $\mathrm{H}^3(\mathrm{X})$ via $A^4_{\mu mnp}(x,y) = \sum_L A^{4L}_\mu(x)\omega^L_{mnp}(y)$, where $\{\omega_L\}_{L=1,\ldots,b_3}$ is a basis of $\mathrm{H}^3(\mathrm{X})$. This is usually written in a symplectic basis $\{\alpha_a, \beta^a\}_{a=0,\ldots,h^{2,1}}$, for which $\int_X \alpha_a \wedge \beta^b = \delta^b_a$, as an expansion of the field strength

$$\mathbf{F}(x,y) = F^a(x) \wedge \alpha_a - G_a(x) \wedge \beta^a. \tag{2}$$

Being a five-form in ten dimensions, the field strength $\mathbf{F}$ admits (anti)self-duality constraints with respect to Hodge duality, $\mathbf{F} = \pm *_{10}\mathbf{F}$. The ten dimensional Hodge operator $*_{10}$ factorizes into a 4D and a 6D part $*_{10} = *_4 *_6$. A solution to the antiselfduality constraint in 10D as well as the Biachi identity $d\mathbf{F} = \mathbf{0}$ can be obtained by setting [18]

$$\mathbf{F} = \mathrm{Re}\left(\mathbf{E} \wedge (\omega^{\mathbf{2,1}} + \omega^{\mathbf{0,3}})\right), \tag{3}$$

where [13, 8]

$$\mathbf{E} \equiv q \sin\theta d\theta \wedge d\phi - iq\frac{e^{2U(r)}}{r^2} dt \wedge dr \tag{4}$$

is a 2-form for which the four-dimensional Hodge duality operator leads to $*_4\mathbf{E} = i\mathbf{E}$.

2.2

The dynamics of a string background configuration can be derived by either reducing the IIB effective action with a small superspace ansatz [12], or via the supersymmetry variation

constraints of the fermions in nontrivial backgrounds, in particular the gravitino and gaugino variations. Defining on $\mathrm{H}^3(\mathrm{X})$ an inner product $< \cdot, \cdot >$ via

$$< \omega, \eta >= \int_X \omega \wedge \eta, \tag{5}$$

the gravitino equation involves the integrated version of the 5-form field strength [2] [6]

$$\mathrm{T}^- = e^{K/2} < \Omega, \mathrm{F}^- >= e^{K/2} \left(\mathcal{G}_a F^{-,a}(x) - z^a G_a^-(x) \right) \tag{6}$$

with the Kähler potential

$$e^{-K} = i < \Omega, \bar{\Omega} >= -i(z^a \bar{\mathcal{G}}_a - \bar{z}^a \mathcal{G}_a), \tag{7}$$

where the second equation is written in terms of the periods $z^a =< \Omega, \beta^a >= \int_{A^a} \Omega$, and $\mathcal{G}_a =< \Omega, \alpha_a >= \int_{B_a} \Omega$ with respect to a symplectic dual homological basis $\{A^a, B_a\}$, whose dual cohomological basis is denoted by $\{\alpha_a, \beta^a\} \subset \mathrm{H}^3(\mathrm{X})$. The holomorphic three-form thus can be expanded as $\Omega = z^a \alpha_a - \mathcal{G}_a \beta^a$.

The supersymmetry transformation of the gravitino $\psi^A = \psi^A_\mu dx^\mu$ can then be written as

$$\delta\psi^A = D\varepsilon^A + dx^\mu \mathrm{T}^-_{\mu\nu} \gamma^\nu (\epsilon\varepsilon)^{\mathrm{A}}, \tag{8}$$

where γ^μ denote the covariant Dirac matrices. The variation of the gaugino of the abelian multiplets takes the form

$$\delta\lambda^{iA} = i\gamma^\mu \partial_\mu z^i \epsilon^A + \frac{i}{2} G^{-,i}_{\mu\nu} \gamma^{\mu\nu} (\epsilon\varepsilon)^A. \tag{9}$$

2.3

Plugging these ingredients into the supersymmetry transformation behavior of the gravitino and the gaugino fields, and demanding that the vacuum remains fermion free, leads to the following equations for the moduli and the spacetime function $U(r)$

$$\begin{aligned} \frac{dU}{d\rho} &= -e^U |Z| \\ \frac{dz^i}{d\rho} &= -2e^U g^{i\bar{j}} \partial_{\bar{j}} |Z|, \end{aligned} \tag{10}$$

where

$$Z(\Gamma) = e^{K/2} \int_\Gamma \Omega = e^{K/2} \int_X \eta_\Gamma \wedge \Omega \tag{11}$$

is the central charge of the cycle $\Gamma \in \mathrm{H}_3(\mathrm{X})$ with Poincare dual $\eta_\Gamma \in \mathrm{H}^3(\mathrm{X})$ and $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ is the metric derived from the Kähler potential K.

The fixed point condition of the attractor equation can be rewritten in a geometrical way as the Hodge condition

$$\mathrm{H}^3(\mathrm{X}, \mathbb{Z}) \ni \omega = \omega^{3,0} + \omega^{0,3}. \tag{12}$$

Writing $\omega^{3,0} = -i\bar{C}\Omega$ this can also be formulated as

$$\begin{aligned} ip^a &= \bar{C} z^a - C\bar{z}^a \\ iq_a &= \bar{C}\mathcal{G}_a - C\bar{\mathcal{G}}_a, \end{aligned} \tag{13}$$

where $C = e^{K/2}Z$. This system describes a set of $b_3(X)$ charges (p^a, q_a) determined by the physical 4-dimensional input which in turn determines the system of complex periods of the Calabi-Yau variety. Hence the equations should be solvable. The interesting structure of the fixed point which emerges is that the central charges are determined completely in terms of the charges of the four-dimensional theory. As a consequence the 4D geometry is such that the horizon is a moduli independent quantity. This is precisely as expected because the black hole entropy should not depend on adiabatic changes of the environment [16].

2.4

In reference [18] it is noted that two types of solutions of the attractor equations have particularly interesting properties. The first of these is provided by the triple product of a torus, while the second is a product of a K3 surface and a torus. Both solutions are special in the sense that they involve elliptic curves. In the case of the product threefold $X = \mathrm{K3} \times \mathrm{E}$ the simplifying feature is that via Künneth's theorem one finds $\mathrm{H}^3(\mathrm{K3} \times \mathrm{E}) \cong \mathrm{H}^2(\mathrm{K3}) \otimes \mathrm{H}^1(\mathrm{E})$, and therefore the cohomology group of the threefold in the middle dimension is isomorphic to two copies of the cohomology group $\mathrm{H}^2(\mathrm{K3})$. The attractor equations for such threefolds have been considered in [1]. The resulting constraints determine the holomorphic form of both

factors in terms of the charges (p,q) of the fields. Moore finds that the complex structure τ of the elliptic curve $E = \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ is solved as

$$\tau_{p,q} = \frac{p \cdot q + \sqrt{D_{p,q}}}{p^2}, \tag{14}$$

where $D_{p,q} = (p \cdot q)^2 - p^2 q^2$ is the discriminant of a BPS state labelled by

$$\omega = (p,q) \in \mathrm{H}^3(\mathrm{K3} \times \mathrm{E}, \mathbb{Z}). \tag{15}$$

The holomorphic two form on K3 is determined as $\Omega^{2,0} = \mathcal{C}(q - \bar{\tau}p)$, where $\mathcal{C}$ is a constant. Moore makes the interesting observation that this result is known to imply that the elliptic curve determined by the attractor equation is distinguished by exhibiting a particularly symmetric structure, i.e. that the endomorphism algebra End(E) is enlarged. In general End(E) is just the ring $\mathbb{Z}$ of rational integers. For special curves however there are two other possibilities for which End(E) is either an order of a quadratic imaginary field, or it is a maximal order in a quaternion algebra. The latter possibility can occur only when the field K over which E is defined has positive characteristic. Elliptic curves are said to admit complex multiplication if the endomorphism algebra is strictly larger than the ring of rational integers.

2.5

The important point here is that the property of complex multiplication appears if and only if the j-invariant $j(\tau)$ is an algebraic integer. This happens if and only if the modulus τ is an imaginary quadratic number. The j-invariant of the elliptic curve E_τ can be defined in terms of the Eisenstein series

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ m,n \text{ coprime}}} \frac{1}{(m\tau + n)^k} \tag{16}$$

as

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}, \tag{17}$$

where $1728\Delta(\tau) = E_4(\tau)^3 - E_6(\tau)^2$. In general the j-function does not take algebraic values, not to mention values in an imaginary quadratic field. We therefore see that in this elliptic

setting the solutions of the attractor equations can be characterized as varieties which admit complex multiplication and determine a quadratic imaginary field $K_D = \mathbb{Q}(i\sqrt{|D|})$.

Once this is recognized several classical results about elliptic curves with complex multiplication are available to illuminate the nature of the attractor variety. One of these results is that the extension $K_D(j(\tau))$ obtained by adjoining the j-value to K_D is the Hilbert class field. Geometrically there is a Weierstrass model, i.e. a projective embedding of the elliptic curve of the form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \tag{18}$$

that is defined over this extension $K_D(j(\tau))$.

Even more interesting is that it is possible to construct from the geometry of the elliptic curve the maximal abelian extension of K_D by considering the torsion points E_{tor} on the curve E, i.e. points of finite order with respect to the group law. To do this consider the Weber function ϕ_E on the curve E. Assuming that the characteristic of the field K_D is different from 2 or 3, the elliptic curve can be embedded via the simplified Weierstrass form

$$y^2 = x^3 + Ax + B \tag{19}$$

with discriminant

$$\Delta = -16(4A^3 + 27B^2) \neq 0. \tag{20}$$

The Weber function can then be defined as

$$\phi_E(p) = \left\{ \begin{array}{ll} \frac{AB}{\Delta} x(p) & \text{if } j(E) \neq 0 \text{ or } 1728 \\ \frac{A^2}{\Delta} x^2(p) & \text{if } j(E) = 1728 \\ \frac{B}{\Delta} x^3(p) & \text{if } j(E) = 0. \end{array} \right\} \tag{21}$$

and the Hilbert class field $K_D(j(\tau))$ can be extended to the full maximal abelian extension K_D^{ab} of K_D by adjoining the Weber values of the torsion points $K_D^{\mathrm{ab}} = K_D(j(\tau), \{\phi_E(T) \mid T \in \mathrm{E}_{\mathrm{tor}}\})$. We see from this that the attractor equations pick out special elliptic curves which lead to a rich arithmetic structure. It is this set of tools which we wish to generalize to the framework of Calabi-Yau varieties proper, i.e. those with finite fundamental group.

3 Deligne's Period Conjecture

3.1

In this section we briefly review Deligne's conjecture in its motivic formulation. This is useful because it will allow us to provide a general perspective for our results which will furnish a useful general framework in which to investigate the arithmetic nature of attractor varieties. Motives are somewhat complicated objects whose status is reminiscent to string theory: different realizations are used to probe what is believed to be some yet unknown unifying universal cohomology theory of varieties. More precisely, motives are characterized by a triplet of different cohomology theories together with a pair of compatibility homomorphisms. In terms of these ingredients a motive then is described by the quintuplet of objects

$$(\mathrm{M_B}, \mathrm{M_{dR}}, \mathrm{M_\ell}, \mathrm{I_{B,\sigma}}, \mathrm{I_{\ell,\bar{\sigma}}}), \tag{22}$$

where the three first entries are cohomology objects constructed via Tate twists from the Betti, de Rham, and étale cohomology, respectively. Furthermore $\mathrm{I_{B,\sigma}}$ describes a map between the Betti and de Rham cohomology, while $\mathrm{I_{\ell,\bar{\sigma}}}$ is a map between Betti and étale cohomology[2]. In the following we will focus mostly on motives derived from the first (co)homology groups $\mathrm{H^1(A)}$ and $\mathrm{H_1(A)}$ of abelian varieties A, as well as the primitive cohomology of Fermat hypersurfaces.

3.2

The second ingredient in Deligne's conjecture is the concept of a geometric $L-$function. This can be described in a number of equivalent ways. Conceptually the perhaps simplest approach results when it can be derived via Artin's zeta function as the Hasse-Weil L-function induced by the underlying variety, i.e. a way of counting solutions of the variety over finite fields. The complete L-function receives contributions from two fundamentally different factors $\Lambda(\mathrm{M,s}) = \mathrm{L_\infty(M,s)L(M,s)}$. The infinity term $L_\infty(M,s)$ originates from those fields over which the underlying variety has bad reduction, i.e. it is singular, while the second term $L(\mathrm{M,s})$ collects all the information obtained from the finite fields over which the variety is smooth. The

[2]Detailed reviews of motives can be found in [15].

complete L-function is in general expected to satisfy a functional equation, relating its values at s and $1-s$. A motive is called critical if neither of the infinity factors in the functional equation has a pole at $s=0$.

3.3

The final ingredient is the concept of the period of a motive, a generalization of ordinary periods of varieties. Viewing the motive M as a generalized cohomology theory Deligne formulates the notion of a period $c^+(M) \in \mathbb{C}^*/\mathbb{Q}^*$ by taking the determinant of a compatibility homomorphism

$$I_{\mathrm{B},\sigma} : \mathrm{M_B} \longrightarrow \mathrm{M_{dR}} \tag{23}$$

between the Betti and the deRham realizations of the motive M. Deligne's basic conjecture then relates the period and the L-function via $L(M,0)/c^+(M) \in \mathbb{Q}$. Contact with the Hasse-Weil L-function is made by noting that for motives of the type $M = \mathrm{H(X)(m)}$ with Tate twists one has $L(\mathrm{M},0) = \mathrm{L(X,m)}$.

3.4

Important for us is a generalization of this conjecture that involves motives with coefficients in a field E. Such motives can best be described via algebraic Hecke characters which are of particular interest for us because they come up in the L-function of projective Fermat varieties. Algebraic Hecke characters were first introduced by Weil as Hecke characters of type A_0, which is what they are called in the older literature. In the context of motives constructed from these characters the field E becomes the field of complex multiplication. In this more general context Deligne's conjecture says that the L-function of the motive and the period take values in the same field, the CM field of the motive.

Deligne Period Conjecture:

$$\frac{L(M,0)}{c^+(M)} \in E. \tag{24}$$

For Fermat hypersurfaces Deligne's conjecture is in fact a theorem, proven by Blasius [3].

4 Abelian Varieties from Brieskorn-Pham Hypersurfaces

4.1

An abelian variety over some number field K is a smooth, geometrically connected, projective variety which is also an algebraic group with the group law $A \times A \longrightarrow A$ defined over K. A concrete way to construct abelian varieties is via complex tori $\mathbb{C}^n/\Lambda$ with respect some lattice Λ that is not necessarily integral and admits a Riemann form. The latter is defined as an $\mathbb{R}$-bilinear form $<,>$ on $\mathbb{C}^n$ such that $< x, y >$ takes integral values for all $x, y \in \Lambda$, $< x, y >= - < y, x >$, and $< x, iy >$ is a positive symmetric form, not necessarily non-degenerate. Then one has the result that a complex torus $\mathbb{C}^n/\Lambda$ has the structure of an abelian variety if and only if there exists a non-degenerate Riemann form on $\mathbb{C}^n/\Lambda$.

4.2

A special class of abelian varieties are those of CM-type, so-called complex multiplication type. The reason for these varieties to be of particular interest is that certain number theoretic question can be addressed in a systematic fashion for this class. Consider a number field K over the rational numbers $\mathbb{Q}$ and denote by $[K : \mathbb{Q}]$ the degree of the field K over $\mathbb{Q}$, i.e. the dimension of K over the subfield $\mathbb{Q}$. An abelian variety A of dimension n is called a CM$-$variety if there exists an algebraic number field K of degree $[K : \mathbb{Q}] = 2n$ over the rationals $\mathbb{Q}$ which can be embedded into the endomorphism algebra $\mathrm{End(A)} \otimes \mathbb{Q}$ of the variety. More precisely, a CM-variety is a pair (A, θ) with $\theta : K \longrightarrow \mathrm{End(A)} \otimes \mathbb{Q}$ an embedding of K. It follows from this that the field K necessarily is a CM field, i.e. a totally imaginary quadratic extension of a totally real field. The important ingredient here is that the restriction to $\theta(K) \subset \mathrm{End(A)} \otimes \mathbb{Q}$ is equivalent to the direct sum of n isomorphisms $\varphi_1, ..., \varphi_n \in \mathrm{Iso(K, \mathbb{C})}$ such that $\mathrm{Iso(K, \mathbb{C})} = \{\varphi_1, ..., \varphi_\mathrm{n}, \rho\varphi_1, ..., \rho\varphi_\mathrm{n}\}$, where ρ denotes complex conjugation. These considerations lead to the definition of calling the pair $(K, \{\varphi_i\})$ a CM-type, in the present context, the CM-type of a CM-variety (A, θ).

4.3

The context in which these concepts will appear below is provided by varieties which have complex multiplication by a cyclotomic field $K = \mathbb{Q}(\mu_n)$, where μ_n denotes the cyclic group generated by a nontrivial n'th root of unity ξ_n. The degree of $\mathbb{Q}(\mu_n)$ is given by $[\mathbb{Q}(\mu_n) : \mathbb{Q}] = \phi(n)$, where $\phi(n) = \#\{m \in \mathbb{N} \mid m < n,\ \mathrm{gcd(m,n)} = 1\}$ is the Euler function. Hence the abelian varieties encountered below will have complex dimension $\phi(n)/2$.

In the following we first reduce the cohomology of the Brieskorn-Pham varieties to that generated by curves and then analyze the structure of the resulting weighted curve Jacobians.

4.4 Curves and the cohomology of threefolds

The difficulty of general higher dimensional varieties is that there is no immediate way to recover abelian varieties, thus making it non-obvious how to generalize the concept of complex multiplication from elliptic curves with complex multiplication, which are abelian varieties. As a first step we need to disentangle the Jacobian of the elliptic curve from the curve itself. This would lead us to the concept of the middle-dimensional cohomology, more precisely the intermediate (Griffiths) Jacobian which is the appropriate generalization of the Jacobian of complex curves. The problem with this intermediate Jacobian is that it is not, in general, an abelian variety.

We will show now that it is possible nevertheless to recover abelian varieties as the basic building blocks of the intermediate cohomology in the case of weighted projective hypersurfaces. The basic reason for this is that the cohomology $\mathrm{H}^3(\mathrm{X})$ for these varieties decomposes into the monomial part and the part coming from the resolution. The monomial part of the intermediate cohomology can easily be obtained from the cohomology of a projective hypersurface of the same degree by realizing the weighted projective space as a quotient variety with respect to a product of discrete groups determined by the weights of the coordinates. For projective varieties $\mathbb{P}_n[d]$ it was shown in [22] that the intermediate cohomology can be determined by

lower-dimensional varieties in combination with Tate twists. Denote the Tate twist by

$$\mathrm{H^i(X)(j) := H^i(X) \otimes W^{\otimes j}} \tag{25}$$

with $W = \mathrm{H}^2(\mathbb{P}_1)$ and let X_d^{r+s} be a Fermat variety of degree d and dimension $r+s$. Then

$$\begin{aligned} \mathrm{H^{r+s}(X_d^{r+s})} \quad \oplus \quad & \sum_{j=1}^{r} \mathrm{H^{r+s-2j}(X_d^{r-1})(j)} \oplus \sum_{\mathrm{k=1}}^{\mathrm{s}} \mathrm{H^{r+s-2k}(X_d^{s-1})(k)} \\ & \cong \mathrm{H^{r+s}(X_d^r \times X_d^s)^{\mu_d} \oplus H^{r+s-2}(X_d^{r-1} \times X_d^{s-1})(1)}. \end{aligned} \tag{26}$$

Here μ_d is the cyclic group of order d which acts on the individual factors as

$$[(x_0, ..., x_r), (y_0, ..., y_s)] \mapsto [(x_0, ..., x_{r-1}, \xi x_r), (y_0, ..., y_{s-1}, \xi y_s)] \tag{27}$$

and induces an action on the hypersurfaces.

4.5

This leaves only the part of the intermediate cohomology of the weighted hypersurface that originates from the resolution. It was shown in [5] that the only singular sets on arbitrary weighted hypersurface Calabi-Yau threefolds are either points or curves. The resolution of singular points contributes to the even cohomology group $\mathrm{H}^2(\mathrm{X})$ of the variety, but does not contribute to the middle-dimensional cohomology group $H^3(X)$. Hence we need to be concerned only with the resolution of curves. This can be described for general CY hypersurface threefolds as follows. If a discrete symmetry group $\mathbb{Z}/n\mathbb{Z}$ of order n acting on the threefold leaves a curve invariant then the normal bundle has fibres $\mathbb{C}_2$ and the discrete group induces an action on these fibres which can be described by a matrix

$$\begin{pmatrix} \alpha^{mq} & 0 \\ 0 & \alpha^m \end{pmatrix},$$

where α is an n'th root of unity and (q, n) have no common divisor. The quotient $\mathbb{C}_2/(\mathbb{Z}/n\mathbb{Z})$ by this action has an isolated singularity which can be described as the singular set of the surface in $\mathbb{C}_3$ given by the equation

$$S = \{(z_1, z_2, z_3) \in \mathbb{C}_3 \mid z_3^n = z_1 z_2^{n-q}\}. \tag{28}$$

The resolution of such a singularity is completely determined by the type (n, q) of the action by computing the continued fraction of $\frac{n}{q}$

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_s}}} \equiv [b_1, ..., b_s]. \tag{29}$$

The numbers b_i specify completely the plumbing process that replaces the singularity and in particular determine the additional generator to the cohomology $\mathrm{H}^*(\mathrm{X})$ because the number of $\mathbb{P}_1$s introduced in this process is precisely the number of steps needed in the evaluation of $\frac{n}{q} = [b_1, ..., b_s]$. This can be traced to the fact that the singularity is resolved by a bundle which is constructed out of $s + 1$ patches with s transition functions that are specified by the numbers b_i. Each of these glueing steps introduces a sphere, which in turn supports a (1,1)-form. The intersection properties of these 2-spheres are described by Hirzebruch-Jung trees, which for a $\mathbb{Z}/n\mathbb{Z}$ action is just an $\mathrm{SU}(n + 1)$ Dynkin diagram, while the numbers b_i describe the intersection numbers. We see from this that the resolution of a curve of genus g thus introduces s additional generators to the second cohomology group $\mathrm{H}^2(\mathrm{X})$, and $g \times s$ generators to the intermediate cohomology $\mathrm{H}^3(\mathrm{X})$.

Hence we have shown that the cohomology of weighted hypersurfaces is determined completely by the cohomology of curves. Since the Jacobian, which we will describe in the next subsection, is the only motivic invariant of a smooth projective curve this says that for weighted hypersurfaces the main motivic structure is carried by their embedded curves.

4.6 Cohomology of weighted curves

For smooth algebraic curves C of genus g the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(\mathrm{C})$ decomposes (over the complex number $\mathbb{C}$) as

$$\mathrm{H}^1_{\mathrm{dR}}(\mathrm{C}) \;\cong\; \mathrm{H}^0(\mathrm{C}, \Omega^1) \oplus \mathrm{H}^1(\mathrm{C}, \mathcal{O}). \tag{30}$$

The Jacobian $J(C)$ of a curve C of genus g can be identified with $J(C) = \mathbb{C}^g/\Lambda$, where Λ is the period lattice

$$\Lambda := \left\{ (\ldots, \int_a \omega_i, \ldots) \mid a \in \mathrm{H}_1(\mathrm{C}, \mathbb{Z}),\; \omega_{\mathrm{i}} \in \mathrm{H}^1(\mathrm{C}) \right\}, \tag{31}$$

where the ω_i form a basis. Given a fixed point $p \in C$ on the curve there is a canonical map from the curve to the Jacobian defined as

$$\varphi_0 : C \longrightarrow J(C) \tag{32}$$

via

$$p \mapsto \left(\ldots, \int_{p_0}^{p} \omega_{r,s}, \ldots \right) \text{ mod } \Lambda. \tag{33}$$

We are interested in curves of Brieskorn-Pham type, i.e. curves of the form

$$\mathbb{P}_{(1,k,\ell)}[d] \ni \left\{ x^d + y^a + z^b = 0 \right\}, \tag{34}$$

such that $a = d/k$ and $b = d/\ell$ are positive rational integers. Without loss of generality we can assume that $(k, \ell) = 1$. We claim that for smooth elements in $\mathbb{P}_{(1,k,\ell)}[d]$ the set of forms

$$\begin{aligned} &\Omega(\mathbb{P}_{(1,k,\ell)}[d]) \\ &\quad = \left\{ \omega_{rst} = y^{s-1} z^{t-d/\ell} dy \mid r + ks + \ell t = 0 \text{ mod } d, \ \left(\begin{array}{c} 1 \leq r \leq d-1, \\ 1 \leq s \leq \frac{d}{k} - 1, \\ 1 \leq t \leq \frac{d}{\ell} - 1 \end{array} \right) \right\} \end{aligned} \tag{35}$$

defines a basis for the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}_{(1,\mathrm{k},\ell)}[\mathrm{d}])$.

In order to show this we view the projective space as the quotient with respect to the actions $\mathbb{Z}_k : [0\ 1\ 0]$ and $\mathbb{Z}_\ell : [0\ 0\ 1]$. This allows us to view the weighted curve as the quotient of a pure projective Fermat type curve

$$\mathbb{P}_{(1,k,\ell)}[d] = \mathbb{P}_2[d]/\mathbb{Z}_k \times \mathbb{Z}_\ell : \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]. \tag{36}$$

These weighted curves are smooth and hence their cohomology is determined by considering those forms on the pure projective curve $\mathbb{P}_2[d]$ which are invariant with respect to the group actions. A basis for $\Omega(\mathbb{P}_2[d])$ is given by the set of forms

$$\left\{ \omega_{rst} = y^{s-1} z^{t-d} dy \mid 0 < r, s, t < d, \quad r + s + t = 0 \text{ (mod d)}, \quad \mathrm{r, s, t} \in \mathbb{N} \right\}. \tag{37}$$

Denote the generator of the $\mathbb{Z}_\ell$ action by α and consider the induced action on ω_{rst}

$$\omega_{rst} \mapsto \alpha^s \omega_{rst}. \tag{38}$$

It follows that the only forms that descend to the quotient with respect to $\mathbb{Z}_\ell$ are those for which $r = 0(\mathrm{mod}\ \ell)$. Similarly we denote by β the generator of the action $\mathbb{Z}_k$ and consider the induced action on the forms ω_{rst}

$$\omega_{rst} \mapsto \beta^{t-d}\omega_{rst}. \tag{39}$$

Again we see that the only forms that descend to the quotient are those for which $s = 0(\mathrm{mod\,k})$.

4.7 Abelian Varieties of weighted Jacobians

It was shown by Faddeev [9][4] in the case of Fermat curves that the Jacobian $J(C_d)$ splits into a product of abelian factors

$$J(C_d) \cong \prod_{\mathcal{O}_i \in I/\mathbb{Z}_d^*} A_{\mathcal{O}_i}, \tag{40}$$

where the $\mathcal{O}_i$ are orbits in I of the multiplicative subgroup $\mathbb{Z}_d^*$ of the group $\mathbb{Z}_d \equiv \mathbb{Z}/d\mathbb{Z}$. More precisely it was shown that there is an isogeny

$$i : J(C_d) \mapsto \prod_{\mathcal{O}_i \in I/\mathbb{Z}_d^*} A_{\mathcal{O}_i}, \tag{41}$$

where an isogeny $i : A \to B$ between abelian varieties is defined to be a surjective homomorphism with finite kernel. We adapt this discussion to the weighted case.

Denote the index set of triples (r, s, t) parametrizing all one-forms by $\mathcal{I}$. The cyclic group $\mathbb{Z}_d$ again acts on $\mathcal{I}$ and the multiplicative subgroup $(\mathbb{Z}_d)^*$ produces a set of orbits

$$[(r, s, t)] \in \mathcal{I}/(\mathbb{Z}_d)^*. \tag{42}$$

Each of these orbits leads to an abelian variety $A_{[(r,s,t)]}$ of dimension

$$\mathrm{dim A_{[(r,s,t)]}} = \frac{1}{2}\phi\left(\frac{\mathrm{d}}{\mathrm{gcd(r, ks, \ell t, d)}}\right), \tag{43}$$

and complex multiplication with respect to the field

$$K_{[(r,s,t)]} = \mathbb{Q}(\mu_{d/\mathrm{gcd(r,ks,\ell t,d)}}). \tag{44}$$

[4]More accessible references of the subject are [24] [14].

This leads to an isogeny

$$i : J(\mathbb{P}_{(1,k,\ell)}[d]) \mapsto \prod_{[(r,s,t)]\in\mathcal{I}/\mathbb{Z}_d^*} A_{[(r,s,t)]}. \tag{45}$$

The complex multiplication type of the abelian factors $A_{[(r,s,t)]}$ of the Jacobian $J(C)$ can be identified with the set

$$\mathrm{H_{rst}} := \{\mathrm{a} \in (\mathbb{Z}/\mathrm{d}\mathbb{Z})^* \mid \ <\mathrm{ar}> + <\mathrm{aks}> + <\mathrm{a}\ell\mathrm{t}>= \mathrm{d}\} \tag{46}$$

via a homorphism from $\mathrm{H_{rst}}$ to the Galois group. This leads to the subgroup of the Galois group of the cyclotomic field given by

$$\mathrm{Gal}(\mathbb{Q}(\mu_{\mathrm{rst}})/\mathbb{Q}) \supset \mathcal{H}_{\mathrm{rst}} = \{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{\mathrm{rst}})/\mathbb{Q}) \mid \mathrm{a} \in \mathrm{H_{rst}}\} \tag{47}$$

and the CM type of $(A_{[(r,s,t)]}, \theta_{rst})$ can be given as

$$(\mathbb{Q}(\mu_{rst}), \{\varphi_1, \cdots, \varphi_n\} = \mathcal{H}_{rst}), \tag{48}$$

where $n = \phi(d_{rst})/2$.

5 Summary and Generalizations

We have seen that the concepts used to describe attractor varieties in the context of elliptic compactifications can be generalized to Calabi-Yau varieties with finite fundamental groups. We have mentioned above that the abelian property is neither carried by the variety itself nor the generalized intermediate Jacobian

$$J^n(X) = \mathrm{H^{2n-1}(X_{an}, \mathbb{C})/H^{2n-1}(X_{an}, \mathbb{Z}(n)) + F^nH^{2n-1}(X_{an}, \mathbb{C})}, \tag{49}$$

but by the Jacobians of the curves that are the building blocks of the middle-dimensional cohomology $\mathrm{H^{\dim_{\mathbb{C}}X}(X)}$. These Jacobians themselves do not admit complex multiplication, unlike the situation in the elliptic case, but instead split into different factors which admit different types of complex multiplication, in general. Furthermore the ring class field can be

generalized to be the field of moduli, and we can consider also points on the abelian variety that are of finite order, i.e. torsion points, and the field extensions they generate.

This allows us to answer a question posed in [18] which asked whether the absolute Galois group $\mathrm{Gal}(\bar{\mathrm{K}}/\mathrm{K})$ could play a role in the context of $N = 2$ compactifications of type IIB strings. This is indeed the case. Suppose we have given an abelian variety A defined over a field K with complex multiplication by a field E. Then there is an action of the absolute Galois group $\mathrm{Gal}(\bar{\mathrm{K}}/\mathrm{K})$ of the closure $\bar{K}$ of K on the torsion points of A. This action is described by a Hecke character which is associated to the fields (K, E) [21].

We have mentioned already that in general the (Griffiths) intermediate Jacobian is only a torus, not an abelian variety. Even in those cases it is however possible to envision the existence of motives via abelian varieties associated to a variety X. Consider the Chow groups $\mathrm{CH^p(X)}$ of codimension p cycles modulo rational equivalence and denote by $\mathrm{CH^p(X)_{Hom}}$ the subgroup of cycles homologically equivalent to zero. Then there is a homomorphism, called the Abel-Jacobi homomorphism, which embeds $\mathrm{CH^p(X)_{hom}}$ into the intermediate Jacobian

$$\phi : \mathrm{CH^p(X)_{hom}} \longrightarrow \mathrm{J^p(X)}. \tag{50}$$

The image of ϕ on the subgroup $\mathcal{A}^p(X)$ defined by cycles algebraically equivalent to zero does in fact define an abelian variety, even if $J^p(X)$ is not an abelian variety but only a torus [19]. Hence we can ask whether attractor varieties are distinguished by Abel-Jacobi images which admit complex multiplication.

Even more general, we can formulate this question in the framework of motives because of Delign's conjecture. Thinking of motives as universal cohomology theories, it is conceivable that attractor varieties lead to motives in the abelian category with (potential) complex multiplication. The standard cycle class map construction of $\mathrm{CH^p(X)_{hom}}$ is replaced by the first term of a (conjectured) filtration in the resulting K-theory.

Putting everything together we see that the two separate discussions in [18] characterizing toroidal attractor varieties on the one hand, and Calabi-Yau hypersurfaces on the other, are just two aspects of our way of looking at this problem. This is the case precisely because

of Deligne's period conjecture which relates the field of the periods to the field of complex multiplication via the L-function of the variety (or motive). Thus a very pretty unified picture emerges.

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